The asymptotic form of the N soliton solution of the Korteweg-de Vries equation

1972 J. Phys. A: Gen. Phys. 5 L132

(http://iopscience.iop.org/0022-3689/5/12/002)

J. Phys. A: Gen. Phys., Vol. 5, December 1972. Printed in Great Britain. © 1972

LETTER TO THE EDITOR

The asymptotic form of the N soliton solution of the Korteweg–de Vries equation

J D GIBBON and J C EILBECK

Department of Mathematics, UMIST, PO Box 88, Manchester, M60 1QD, UK

MS received 6 November 1972

Abstract. The asymptotic form of Hirota's N soliton solution of the Korteweg–de Vries equation is derived. The phase shifts of the N solitons caused by a general collision are found to be *linear* sums of two soliton terms.

The Korteweg–de Vries (KV) equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{1}$$

was introduced by Korteweg and de Vries (1895) in their approximate theory of water waves. Their equation has also been used in plasma physics (Washimi and Taniuti 1966) and in the study of anharmonic lattices (Zabusky 1967). Zabusky and Kruskal (1965) used numerical studies of the KV equation to show that general solutions of the equation evolved into a series of solitary wave solutions of the type

$$u = \tfrac{1}{2}P^2 \operatorname{sech}^2(\tfrac{1}{2}Px - \tfrac{1}{2}P^3t + \delta). \tag{2}$$

These solitary waves, which were named solitons, had the remarkable property that they passed through each other without breaking up, but with an overall change in the phase shift δ. These solutions were studied further by Gardner *et al* (1967), Miura (1968), Miura *et al* (1968), Lax (1968), Su and Gardner (1969), Gardner (1971), Shih (1971) and Benjamin (1972). Recently Hirota (1971) has given a remarkable exact analytic solution for the collision of N solitons. In this letter we clarify the nature of Hirota's solution by examining its asymptotic form.

Hirota's solution to (1) is as follows:

$$u(x,t) = 2\frac{\partial^2}{\partial x^2}\ln f(x,t) \tag{3}$$

$$f(x,t) = \det|M|. \tag{4}$$

The elements of the $N \times N$ matrix M are given by

$$M_{ij} = \delta_{ij} + \frac{2(P_iP_j)^{1/2}}{P_i + P_j}\exp\{\tfrac{1}{2}(\xi_i + \xi_j)\} \tag{5}$$

$$\xi_i = P_i x - P_i^3 t + \xi_i^0. \tag{6}$$

The P_i and ξ_i^0 are arbitrary constants which determine the amplitude and phase, respectively, of the ith soliton. The P_i are assumed to be all different.

We first consider the asymptotic form of (3)–(6) as $t \to -T$, where T is large enough for all the solitons to be well separated. We need the diagonal expansion for f (Hirota 1971, Rubinstein 1970)

$$f = 1 + \sum_{n=1}^{N} \sum_{{}_N C_n} a(i_1, i_2, i_3, \ldots, i_n) \exp(\xi_{i_1} + \xi_{i_2} + \ldots \xi_{i_n}) \tag{7}$$

where

$$a(i_1, i_2, \ldots, i_n) = \prod_{k<l}^{(n)} a(i_k, i_l) \tag{8}$$

$$a(i_k, i_l) = \frac{(P_{i_k} - P_{i_l})^2}{(P_{i_k} + P_{i_l})^2}. \tag{9}$$

Without loss of generality we assume the N solitons are ordered such that $P_1^2 > P_2^2 > \ldots > P_N^2$. For simplicity we assume all the P_i are positive. As $t \to -T$ we examine the solution for values of x such that $\xi_n \approx 0$; $\xi_1, \xi_2, \ldots \xi_{n-1} \to -\mathrm{O}(T)$ and $\xi_{n+1}, \ldots, \xi_N \to +\mathrm{O}(T)$. The next step is to factorize (7) in the form

$$f = \exp(\xi_{n+1} + \ldots + \xi_N) f_n \tag{10}$$

where

$$f_n = A \exp(\xi_1 + \xi_2 + \ldots + \xi_n) + \ldots + B \exp(\xi_n) + C + \ldots + \exp(-\xi_{n+1} - \ldots - \xi_N). \tag{11}$$

The coefficients A, B, C and others in this expansion depend on the P_i only. If we insert (10) into (3) the exponential term $\exp(\xi_{n+1} + \ldots + \xi_N)$ will not contribute since the ξ_i are *linear* functions of x. In the limit $t \to -T$ all the terms in (11) are negligible except for two terms:

$$f_n = B \exp(\xi_n) + C. \tag{12}$$

We define

$$\gamma_n^- = \tfrac{1}{2} \ln(B/C). \tag{13}$$

From (7)–(9) it is easy to show that

$$\gamma_n^- = \tfrac{1}{2} \ln\left(\prod_{i=n+1}^{N} a_{in} \right) = \tfrac{1}{2} \sum_{i=n+1}^{N} \ln(a_{in}) \tag{14}$$

with the a_{in} as defined in (9). We can now write f_n as

$$f_n = 2C \exp(\tfrac{1}{2}\xi_n + \gamma_n^-) \cosh(\tfrac{1}{2}\xi_n + \gamma_n^-) \tag{15}$$

from (3) we have

$$u_n = \tfrac{1}{2} P_n^2 \operatorname{sech}^2(\theta_n + \gamma_n^-) \tag{16}$$

where $\theta_n = \frac{1}{2}\xi_n$. This is the nth soliton before collision. Since all the solitons are well separated at $t = -T$, by assumption, the full amplitude will be simply a linear sum of solitons

$$u = \sum_{n=1}^{N} \tfrac{1}{2} P_n^2 \operatorname{sech}^2(\theta_n + \gamma_n^-). \tag{17}$$

In a similar manner we can calculate the asymptotic form as $t \to +T$:

$$u = \sum_{n=1}^{N} \tfrac{1}{2} P_n^2 \operatorname{sech}^2(\theta_n + \gamma_n^+) \tag{18}$$

where

$$\gamma_n^+ = \tfrac{1}{2} \sum_{i=1}^{n-1} \ln(a_{in}). \tag{19}$$

The total phase change of the nth soliton during collision is

$$\gamma_n^+ - \gamma_n^- = \tfrac{1}{2} \sum_{i=1}^{n-1} \ln(a_{in}) - \tfrac{1}{2} \sum_{i=n+1}^{N} \ln(a_{in}). \tag{20}$$

A physical interpretation is that if the ith soliton *overtakes* the nth soliton it contributes $+a_{in}$ to the phase of the nth soliton: if the ith soliton is *overtaken* by the nth soliton it contributes $-a_{in}$ to the phase of the nth soliton. The remarkable feature of (20) is that it is a *linear* sum of two soliton terms even when three or more solitons collide simultaneously.

Recently an N soliton solution has been proposed (Gibbon and Eilbeck 1972) for an equation with many properties in common with the KV equation, the sine-Gordon equation (Barone *et al* 1971). It is interesting to note that this N soliton solution has a similar linear sum of two soliton terms in the phases of the solitons in the asymptotic limits. The two-soliton phase shift in the sine-Gordon solution is exactly twice that in Hirota's KV solution.

In view of the connection between the sine-Gordon equation and the equations of nonlinear optics (Lamb 1971), it is natural to ask whether there exists a KV analogue of the so-called 0π pulse of nonlinear optics. The 0π pulse is formed from the two-soliton sine-Gordon solution by taking the two-soliton amplitudes to be an anti-hermitian pair of complex numbers. Unfortunately in the KV case this choice gives a complex rather than a real pulse. Taking an hermitian pair in the KV case gives a real pulse, but this pulse is unbounded in the (x, t) plane and cannot be considered as a physical solution of the KV equation.

We would like to thank Dr D M Griffel of the University of Bristol for stimulating our interest in the KV equation, and Dr P J Caudrey of UMIST for several useful conversations. One of us (JCE) would like to acknowledge the financial support of an SRC assistantship under the supervision of Dr R K Bullough.

References

Barone A, Esposito F, Magee C J and Scott A C 1971 *Riv. Nuovo Cim.* **1** 227–67
Benjamin T B 1972 *Proc. R. Soc.* A **328** 153–83
Gardner C S 1971 *J. math. Phys.* **12** 1548–51
Gardner C S, Green J M, Kruskal M D and Miura R M 1967 *Rev. Phys. Lett.* **19** 1095–7
Gibbon J D and Eilbeck J C 1972 *J. Phys.* A: *Gen. Phys.* **5** 122–4
Hirota R 1971 *Phys. Rev. Lett.* **27** 1192–4
Korteweg D J and de Vries G 1895 *Phil. Mag.* **39** 422–43
Lamb G L Jr 1971 *Rev. mod. Phys.* **43** 99–124
Lax P D 1968 *Commun. pure appl. Math.* **21** 467–90

Miura R M 1968 *J. math. Phys.* **9** 1202–4
Miura R M, Gardner G S and Kruskal M D 1968 *J. math. Phys.* **9** 1204–9
Rubinstein J 1970 *J. math. Phys.* **11** 258–66
Shih L Y 1971 *J. math. Phys.* **12** 1735–43
Su C H and Gardner C S 1969 *J. math. Phys.* **10** 536–9
Washimi H and Taniuti T 1966 *Phys. Rev. Lett.* **17** 996–8
Zabusky N J 1967 *Proc. Symp. on Nonlinear Partial Differential Equations* ed W Ames (New York: Academic Press) pp 223–58
Zabusky N J and Kruskal M D 1965 *Phys. Rev. Lett.* **15** 240–3